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On resonant nonlinear bubble oscillations

By J. E. FFOWCS WILLIAMS AND Y. P. GUO
Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, UK

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If a bubble were produced with an initial surface distortion, the energy carried by surface modes could be converted to other modes by nonlinear interaction, a conversion that provides a possible mechanism of second generation by bubbles. Longuet-Higgins (1989*a*, *b*) has argued that volume pulsation would be excited at twice the frequency of the distortion mode and that the response to such excitation is 'surprisingly large' when its frequency is close to the natural resonance frequency of the volumetrical mode. It is shown in this paper that this is feasible only if the driving system is sufficiently energetic to supply the energy involved in those volume pulsations, and that this is not generally the case. In the absence of external sources, the sum of energies in the interacting modes cannot exceed the initial bubble energy; an increase in one mode is always accompanied by a decrease in another. In contrast to any expectation of significant pulsations near resonance, we find that, once modal coupling is admitted, the volumetrical pulsation has very small amplitude in comparison with that of the initial surface distortion. This is because of the constraint of energy, a constraint that becomes more severe once damping is admitted. Our conclusion therefore is that the distortion modes of a bubble are unlikely to be the origin of an acoustically significant bubble response.

1. Introduction

On account of recent interest in underwater sound generation by mechanisms associated with ocean surface motions, bubble oscillations, as a potential source of oceanic ambient noise, have attracted much research in recent years (see, for example, the proceedings of the NATO workshop on natural mechanisms of surface-generated noise in the ocean, held at Lerici, Italy, June 1987). Longuet-Higgins (1989*a*, *b*) has considered the excitation of bubble volume pulsations by nonlinear interactions of bubble surface distortions and concluded that such excitations may be significant if the surface distortion frequency is close to half the pulsation frequency, which provides a possible mechanism for the conversion of purely surface distortion energy into radiative acoustic energy.

Longuet-Higgins uses a direct perturbation scheme that predicts damping-limited volume pulsations near the resonance condition. Though these results point to the possibility of significant volume pulsations, they are not conclusive; the solution is obviously unacceptable if the initial motion is called upon to give up more than its total energy. Initially, in order to establish clear bounds, we do a definite calculation by postulating an initial condition in which only the surface mode is active, and consider what the bubble's subsequent behaviour would be if the distortion mode amplitude were maintained. Without damping the volumetric mode starts to grow linearly with time and the energy contained in this mode grows in proportion to the square of time at resonance. The energy carried by the bubble cannot exceed its initial level; this sets a clear upper bound for the energy that may be transferred to the volumetric mode. The instant at which the growing pulsation energy reaches the level of the initial bubble energy then gives the maximum time for which the perturbation theory could possibly be relevant. We find that this maximum time is only about 4–5 periods of bubble oscillation.

In this direct perturbation scheme, the failure to account for the long-term behaviour of the bubble oscillations results from the failure to account completely for the nonlinear coupling between the interacting modes. When a surface distortion mode is coupled to the volumetric mode, the surface mode cannot maintain a constant amplitude; it energizes the induced volumetric mode so that its own amplitude decays as the volume pulsation grows. Thus, even without any damping, the excited volumetric mode must be of strictly limited amplitude; coupling between two modes limits the amplitudes of both. This class of phenomenon has been extensively studied (e.g. Nayfeh & Mook 1979). One approach capable of accounting for the long-term coupling is the technique of multiple scales, which is the one we utilize in this paper. To demonstrate this, we admit the damping of the surface mode consequent on the driving of the volumetric mode, but neglect other damping effects. Thus, our model gives an upper bound for the volume pulsations due to nonlinear interactions of surface modes; damping due to acoustic, thermal and viscous effects would dissipate part of the initial bubble energy so that the pulsation amplitude would be further reduced.

When resonance occurs, we find that both the surface distortion and the excited volume pulsation undergo modulation. At exact resonance, only amplitude modulations occur and the modulations are monotonic functions of time; the volume pulsation increases as it draws energy from the surface distortion mode. Near, but not at, resonance, energy is exchanged cyclically between the surface and volumetric modes; the oscillations in this case experience both amplitude and phase modulation. The phase modulation, which results in changes in the oscillation frequencies, has previously been observed and studied for oscillations of liquid drops (e.g. Trinh & Wang 1982; Tsamopoulos & Brown 1983). We shall show that the amplitudes of both modes are always bounded and the sum of the energies in the two modes is always equal to the initial bubble energy; the growing of one mode is accompanied by the decay of the other. We shall show that, in contrast to the direct perturbation scheme's indication of large-amplitude pulsation near resonance, even when damping is ignored the induced volumetric mode has actually very small amplitude in comparison with that of the surface mode that is energizing the pulsation. This is because volume pulsation requires more energy than shape distortion if the two maintain the same amplitude of oscillation. In the case where energy comes from the initial bubble deformation in a purely surface mode, even if that finite amount of energy were all converted into pulsation, the amplitude of the pulsation must be smaller than that of the surface mode. Damping makes it smaller still. For near resonance, we show that the surface distortion never reaches zero amplitude; some of the initial energy always remains in the surface mode.

2. General formulation

We consider the oscillations of a gas bubble embedded in water that is assumed to be incompressible, inviscid and of infinite extent. It is convenient to formulate the problem in terms of non-dimensional variables. To this end, we follow Hall &

Seminara (1980) and choose r_0, the radius of the bubble at equilibrium, as the reference lengthscale, and $(r_0^2\rho_\infty/p_\infty)^{\frac{1}{2}}$ as the reference timescale, where ρ_∞ is the constant density in water and p_∞ denotes the constant pressure at infinity. In the water, the incompressible irrotational motion can be described by the velocity potential that satisfies the Laplace equation

$$\nabla^2\phi = 0, \tag{2.1}$$

where ϕ is normalized by $(r_0^2 p_\infty/\rho_\infty)^{\frac{1}{2}}$. We choose spherical coordinates (r, θ, φ) such that the centre of the bubble coincides with the origin of the coordinate system, so that the bubble surface can be specified as

$$f(r,\theta,\varphi,t) = r-1-\eta(\theta,\varphi,t) = 0,$$

with η denoting the non-dimensional surface deformation and t the non-dimensional time.

On the bubble surface $r = 1+\eta$, the kinematic condition that particles in the surface always remain in it can be written as

$$\frac{\mathrm{D}}{\mathrm{D}t}f(r,\theta,\varphi,t) = 0, \tag{2.2}$$

with $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \nabla\phi\cdot\nabla$ being the total derivative. In terms of the coordinates (r, θ, φ) and by introducing the surface gradient defined by

$$\nabla_{\mathrm{s}} = \left\{\frac{\partial}{\partial\theta}, \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right\},$$

the condition (2.2) becomes

$$\frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial r} + \frac{1}{r^2}\nabla_{\mathrm{s}}\eta\cdot\nabla_{\mathrm{s}}\phi = 0, \tag{2.3}$$

which is to be applied on the bubble surface $r = 1+\eta$.

The dynamic condition on the bubble surface is the balance between the pressure forces acting on both sides of it and the force due to surface tension, which, if viscosity is neglected, assumes the form

$$p_{\mathrm{B}} = p + T\nabla\cdot\boldsymbol{n}_{\mathrm{B}}, \tag{2.4}$$

where p_{B} and p are respectively the pressures, normalized by p_∞, in the bubble and in water, T is the surface tension normalized by $r_0 p_\infty$ and $\boldsymbol{n}_{\mathrm{B}}$ represents the unit normal, pointing towards the water region, to the bubble surface with its total curvature denoted by $\nabla\cdot\boldsymbol{n}_{\mathrm{B}}$ (Lamb 1933). On the water side, the pressure p is given by the Bernoulli equation

$$p = 1 - \frac{\partial\phi}{\partial t} - \tfrac{1}{2}(\nabla\phi)^2, \tag{2.5}$$

where the integration constant is set to equal to the non-dimensional pressure at infinity, because both ϕ and its derivatives vanish as $r\to\infty$. The pressure p_{B} in the bubble can be expressed in terms of the surface deformation η, provided that the gas in the bubble is assumed to undergo adiabatic changes. In this case, p_{B} is related to

$$V_{\mathrm{B}} = \frac{1}{3}\int_0^{\pi}\int_0^{2\pi}(1+\eta)^3\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

which is the instantaneous bubble volume divided by r_0^3, through the static law

$$p_B = (1+2T)\left(\frac{V_0}{V_B}\right)^{\gamma}, \tag{2.6}$$

where γ denotes the ratio of specific heats, $V_0 = \frac{4}{3}\pi$ is the value of V_B when the bubble is at equilibrium and $1+2T$ is the non-dimensional pressure in the bubble at equilibrium. For convenience, we introduce the overbar to denote the average over a unit sphere; for example,

$$\bar{\eta} = \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} \eta(\theta,\varphi)\sin\theta\,d\theta\,d\varphi.$$

Thus, we have

$$\frac{V_B}{V_0} = 1+3\bar{\eta}+3\overline{\eta^2}+\overline{\eta^3},$$

from which the bubble pressure (2.6) can be written in terms of η as

$$p_B = (1+2T)(1+3\bar{\eta}+3\overline{\eta^2}+\overline{\eta^3})^{-\gamma}. \tag{2.7}$$

On substituting (2.5) and (2.7) into the dynamical boundary condition (2.4), it follows that

$$(1+2T)(1+3\bar{\eta}+3\overline{\eta^2}+\overline{\eta^3})^{-\gamma} = 1-\frac{\partial\phi}{\partial t}-\tfrac{1}{2}(\nabla\phi)^2+T\nabla\cdot\boldsymbol{n}_B, \tag{2.8}$$

to be imposed on the bubble surface $r = 1+\eta$.

The fact that both (2.3) and (2.8) are imposed on the oscillating bubble surface $r = 1+\eta$ poses difficulties in solving the problem. However, if the bubble is only slightly deformed, at some initial instant, from its equilibrium spherical shape, it can be expected that the subsequent oscillations caused by such initial deformations will be of small amplitude compared with the bubble radius at equilibrium. In this case, we can expand both conditions from $r = 1+\eta$ to the position of the bubble surface at equilibrium $r = 1$, according to

$$F(1+\eta) = F(1)+\eta\frac{\partial F(1)}{\partial r}+\frac{1}{2!}\eta^2\frac{\partial^2 F(1)}{\partial r^2}+\ldots, \tag{2.9}$$

which is convergent for $|\eta| < 1$.

Applying the expansion to (2.7), the pressure in the bubble becomes

$$p_B = (1+2T)[1-3\gamma\bar{\eta}-3\gamma\overline{\eta^2}+\tfrac{9}{2}\gamma(1+\gamma)\bar{\eta}^2],$$

and it is straightforward to derive (e.g. Longuet-Higgins 1989*a*)

$$\nabla\cdot\boldsymbol{n}_B = 2-(2+\nabla_s^2)\eta+2(\eta^2+\eta\nabla_s^2\eta),$$

where $\nabla_s^2 = \nabla_s\cdot\nabla_s$ and the results have been truncated from the term of order η^3. Since ϕ and its derivatives are all of the same order as η, expansions of (2.3) and (2.8) to the order η^2 are found to be

$$\left.\begin{aligned}\frac{\partial\eta}{\partial t}-\frac{\partial\phi}{\partial r} &= \eta\frac{\partial^2\phi}{\partial r^2}-\nabla_s\eta\cdot\nabla_s\phi,\\ \frac{\partial\phi}{\partial t}+T(2+\nabla_s^2)\eta-3\gamma(1+2T)\bar{\eta} &= 2T(\eta^2+\eta\nabla_s^2\eta)-\eta\frac{\partial^2\phi}{\partial r\,\partial t}-\tfrac{1}{2}(\nabla\phi)^2\\ &\quad+3\gamma(1+2T)[\overline{\eta^2}-\tfrac{3}{2}(1+\gamma)\bar{\eta}^2],\end{aligned}\right\} \tag{2.10}$$

which are now both to be applied at $r = 1$.

To complete the formulation, initial conditions must be specified. Since our concern here is the way in which surface distortion motions are converted into volume pulsations, we can specify the initial conditions, without loss of generality, as

$$\eta = \epsilon S_l(\theta,\varphi) \quad \text{and} \quad \frac{\partial \eta}{\partial t} = 0 \quad \text{at} \quad t = 0, \tag{2.11}$$

where ϵ is a small parameter, introduced here to ensure that the bubble oscillations are of small amplitude so that the Taylor expansion (2.9) applied to the boundary conditions is convergent, and $S_l(\theta,\varphi)$ is the spherical harmonic function of order l (e.g. Lamb 1933; Gradshteyn & Ryzhik 1980). We assume that l is larger than one, the initial deformation being a purely surface distortion.

3. Energy conservation principle

For inviscid bubble oscillations in an incompressible liquid, according to our initial-value calculation all the energy initially lies in the deformation of the bubble surface. The total energy at any instant must be equal to this initial value, though it may be exchanged, as a result of oscillation, from one form to another (from kinetic energy to surface tension energy, for example), and transferred between different modes through nonlinear interactions. It is appropriate to establish a conservation principle which can, on the one hand, give insight to the energetics of the oscillation process without directly solving the problem, and on the other hand, serve as a check on the solutions which we will discuss in the subsequent sections.

The energy equation can be derived by starting with the inviscid non-dimensional momentum equation $\mathrm{D}\boldsymbol{\nabla}\phi/\mathrm{D}t + \boldsymbol{\nabla}p = 0$. By taking the scalar product of this with $\boldsymbol{\nabla}\phi$, transferring it into the differential operators by using (2.1), an energy equation can be derived:

$$\frac{\mathrm{D}}{\mathrm{D}t}[\tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2] + \boldsymbol{\nabla}\cdot(p\boldsymbol{\nabla}\phi) = 0, \tag{3.1}$$

which simply states that the variation in the kinetic energy of a fluid particle, which is also its total energy since compressibility of the water is neglected, is equal to the work done by the pressure acting on it.

Integrating the energy equation (3.1) over the entire water region and transferring the second term into surface integrals through the divergence theorem, we find that

$$\int \frac{\mathrm{D}}{\mathrm{D}t}[\tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2]\,\mathrm{d}^3\boldsymbol{x} + \int_{S_\infty} p\boldsymbol{\nabla}\phi\cdot\boldsymbol{n}_\infty\,\mathrm{d}s_\infty - \int_{S_\mathrm{B}} p\boldsymbol{\nabla}\phi\cdot\boldsymbol{n}_\mathrm{B}\,\mathrm{d}s_\mathrm{B} = 0,$$

where S_∞ denotes the control surface at infinity with unit outwards normal $\boldsymbol{n}_\infty$ and S_B represents the bubble surface. Considering that the pressure at infinity is essentially equal to 1 and that on the bubble surface is given by (2.4), this can be rewritten as

$$\int \frac{\mathrm{D}}{\mathrm{D}t}[\tfrac{1}{2}(\boldsymbol{\nabla}\phi)^2]\,\mathrm{d}^3\boldsymbol{x} + \int_{S_\infty} \boldsymbol{\nabla}\phi\cdot\boldsymbol{n}_\infty\,\mathrm{d}s_\infty - \int_{S_\mathrm{B}} p_\mathrm{B}\boldsymbol{\nabla}\phi\cdot\boldsymbol{n}_\mathrm{B}\,\mathrm{d}s_\mathrm{B} + T\int_{S_\mathrm{B}} \boldsymbol{\nabla}\cdot\boldsymbol{n}_\mathrm{B}\,\boldsymbol{\nabla}\phi\cdot\boldsymbol{n}_\mathrm{B}\,\mathrm{d}s_\mathrm{B} = 0.$$

In the Appendix, it is shown that the four terms in this equation are respectively the rate of change in the total kinetic energy in water, the work done at infinity by p_∞, the internal energy of the bubble and the energy associated with surface tension, all being non-dimensionalized by $r_0^3 p_\infty$. If we denote the four components respectively by E_{K}, E_{W}, E_{B} and E_T, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(E_{\mathrm{K}}+E_{\mathrm{W}}+E_{\mathrm{B}}+E_T) = 0, \tag{3.2}$$

which, since, for the initial-value problem where there is no kinetic energy in the field at the initial instant, all the energy comes from the initial surface deformation, becomes

$$E_{\mathrm{K}}+E_{\mathrm{W}}+E_{\mathrm{B}}+E_T = E_{\mathrm{B}}(0)+E_T(0). \tag{3.3}$$

The initial bubble energy may be distributed among the four components, but their sum is always equal to this initial value. This is, of course, the consequence of neglecting both dissipation and radiation; if these are taken into account, the bubble oscillations will be damped owing to energy losses. Equation (3.3) is derived from the governing equation with fully nonlinear boundary conditions so that E_{K}, E_{W}, E_{B} and E_T are the total energies; they not only contain the excess energy (of second order in the perturbation variables) but also the zeroth-order, first-order and higher-order terms (see the Appendix for the definitions of E_{K}, E_{W}, E_{B} and E_T). Though this way of deriving the energy equation clearly constrains the energy exchange process among different forms, it is also useful and instructive to cast the energy equation solely in terms of the second-order quantities in the perturbation variables, so that the way in which energy is converted between different modes can be made clear.

In the Appendix, it is shown that the four components E_{K}, E_{W}, E_{B} and E_T can all be expressed in terms of the perturbation variables η and ϕ. By making use of (A 16) and (A 17) for E_{B} and E_T, the initial energy in the bubble can be evaluated from the initial condition (2.11), which gives

$$E_T(0)+E_{\mathrm{B}}(0) = \left(4\pi T+\frac{(1+2T)\,V_0}{\gamma-1}\right)+2\pi T\epsilon^2\frac{(l-1)\,(l+2)}{2l+1}, \tag{3.4}$$

where use has been made of the formulae (Gradshteyn & Ryzhik 1980)

$$\overline{S_l^2(\theta,\varphi)} = \frac{1}{2l+1}, \quad \overline{[\nabla_{\mathrm{s}} S_l(\theta,\varphi)]^2} = \frac{l(l+1)}{2l+1}.$$

The terms in the large parentheses on the right-hand side of (3.4) are the sum of the surface tension energy and the internal energy in the bubble at equilibrium state, and the second term is the excess energy over this value at the initial instant, which is the part that energizes the bubble oscillations. On substituting (A 14)–(A 17) into (3.3), with the initial energy given by (3.4), it follows that

$$T\overline{(\nabla_{\mathrm{s}}\eta)^2}-2T\overline{\eta^2}-\overline{\phi(\partial\eta/\partial t)}+3\gamma(1+2T)\,\overline{\eta}^2 = \epsilon^2\frac{(l-1)\,(l+2)\,T}{2l+1}. \tag{3.5}$$

This energy equation only involves second-order quantities; the zeroth- and first-order energies are conserved within themselves.

To demonstrate the energy relations between different modes, we notice that the bubble oscillations can be decomposed into spherical harmonics of the form

$$\eta = \sum_n \epsilon\eta_n(t)\,S_n(\theta,\varphi), \quad \phi = \sum_n \epsilon\phi_n(t)\,S_n(\theta,\varphi)\frac{1}{r^{n+1}}.$$

When these are inserted into (3.5), with the use of the formulae (Gradshteyn & Ryzhik 1980)

$$\overline{S_n(\theta,\varphi)\,S_{n'}(\theta,\varphi)} = \begin{cases} \dfrac{1}{2n+1} & \text{when } n = n' \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\overline{\nabla_s S_n(\theta,\varphi)\cdot\nabla_s S_{n'}(\theta,\varphi)} = \begin{cases} \dfrac{n(n+1)}{2n+1} & \text{when } n = n' \\ 0 & \text{otherwise,} \end{cases}$$

the principle of energy conservation becomes

$$\sum_n E_n(t) = \frac{T(l-1)\,(l+2)}{2l+1}, \tag{3.6}$$

where $E_n(t)$ is the energy contained in the nth mode and is defined by

$$E_n(t) = \frac{1}{2n+1}\left(\frac{\omega_n^2}{n+1}\eta_n^2 - \phi_n\frac{\partial\eta_n}{\partial t}\right), \tag{3.7}$$

with ω_n being the natural frequency of the nth mode in linear oscillation, which is given by (e.g. Lamb 1933 and Strasberg 1956)

$$\omega_n^2 = \begin{cases} 3\gamma(1+2T)-2T & \text{when } n = 0 \\ T(n-1)\,(n+1)\,(n+2) & \text{otherwise.} \end{cases} \tag{3.8}$$

Equation (3.6) reveals the energy relation between different modes; the initial bubble energy which is in the form of surface distortion may be transferred to other modes, which, as will be shown in the following sections, can only occur through resonant nonlinear interactions. The sum of the energies in all the modes in the interaction is always equal to the initial bubble energy.

4. Solution by direct expansion

Since the initial condition (2.11) contains a small parameter ϵ, it is natural to seek solutions η and ϕ in terms of a power series of ϵ. This is the method used by Longuet-Higgins (1989*a*, *b*). In this section, we re-examine this method to reveal its limitations. In this scheme, conventional power series expansions are applied:

$$\left.\begin{aligned} \eta &= \epsilon\eta^{(1)} + \epsilon^2\eta^{(2)} + \ldots, \\ \phi &= \epsilon\phi^{(1)} + \epsilon^2\phi^{(2)} + \ldots. \end{aligned}\right\} \tag{4.1}$$

On substituting these into (2.1) and (2.10), and grouping terms of order ϵ, the first-order problem is found to be precisely the linear oscillation problem, the solutions of which comply with the initial conditions (2.11) (e.g. Lamb 1933):

$$\left.\begin{aligned} \eta^{(1)} &= \cos\omega_l t\, S_l(\theta,\varphi), \\ \phi^{(1)} &= \frac{\omega_l}{l+1}\sin\omega_l t\, S_l(\theta,\varphi)\frac{1}{r^{l+1}}. \end{aligned}\right\} \tag{4.2}$$

The second-order velocity potential $\phi^{(2)}$ also satisfies the Laplace equation, but the boundary conditions relating $\phi^{(2)}$ to $\eta^{(2)}$ are no longer homogeneous. Instead, the nonlinear terms in (2.10) are to be calculated from the first-order solutions $\eta^{(1)}$ and $\phi^{(1)}$. By expressing these nonlinear terms in terms of the spherical harmonic functions, the boundary conditions for $\phi^{(2)}$ and $\eta^{(2)}$ are found to be (for details, see Longuet-Higgins 1989*a*)

$$\left.\begin{aligned} \frac{\partial \eta^{(2)}}{\partial t}-\frac{\partial \phi^{(2)}}{\partial r} &= \sin 2\omega_l t \sum_n \varpi_n S_n(\theta,\varphi), \\ \frac{\partial \phi^{(2)}}{\partial t}+T(2+\nabla_s^2)\,\eta^{(2)}-3\gamma(1+2T)\,\overline{\eta^{(2)}} &= \cos 2\omega_l t \sum_n \beta_n S_n(\theta,\varphi)+\sum_n \mu_n S_n(\theta,\varphi), \end{aligned}\right\} \tag{4.3}$$

where the sum is from zero to $2l$ and contains only even-order terms because ϖ_n, β_n and μ_n vanish for odd n, according to their definitions (see, for example, Gradshteyn & Ryzhik 1980)

$$\left.\begin{aligned} \varpi_n &= \frac{2n+1}{2}\left[(l+2)\,\overline{S_l^2 S_n}-\frac{1}{l+1}\overline{(\nabla_s S_l)^2 S_n}\right]\omega_l, \\ \beta_n &= \frac{2n+1}{4}\left[T(3l^3+2l^2-7l-2)\,\overline{S_l^2 S_n}+\frac{\omega_l^2}{(l+1)^2}\overline{(\nabla_s S_l)^2 S_n}+\frac{6\gamma(1+2T)}{2l+1}\overline{S_n}\right], \\ \mu_n &= \frac{2n+1}{4}\left[T(l^3-2l^2-5l+2)\,\overline{S_l^2 S_n}-\frac{\omega_l^2}{(l+1)^2}\overline{(\nabla_s S_l)^2 S_n}+\frac{6\gamma(1+2T)}{2l+1}\overline{S_n}\right]. \end{aligned}\right\} \tag{4.4}$$

Since the initial deformation (2.11) is of order ϵ, the second-order initial conditions are simply

$$\eta^{(2)} = \frac{\partial \eta^{(2)}}{\partial t} = 0 \quad \text{at} \quad t=0. \tag{4.5}$$

Considering the boundary condition (4.3) and the initial condition (4.5), the solutions for $\phi^{(2)}$ and $\eta^{(2)}$ can be assumed to be of the form

$$\left.\begin{aligned} \eta^{(2)} &= \sum_n \eta_n^{(2)}(t)\,S_n(\theta,\varphi), \\ \phi^{(2)} &= \sum_n \phi_n^{(2)}(t)\,S_n(\theta,\varphi)\frac{1}{r^{n+1}}, \end{aligned}\right\} \tag{4.6}$$

which, on substituting into (4.3), leads to an equation for $\eta_n^{(2)}$:

$$\frac{\mathrm{d}^2\eta_n^{(2)}}{\mathrm{d}t^2}+\omega_n^2\,\eta_n^{(2)} = [2\omega_l\varpi_n-(n+1)\,\beta_n]\cos 2\omega_l t-(n+1)\,\mu_n,$$

with ω_n and ω_l given by (3.8). The solution to this subject to the initial condition (4.5) is

$$\eta_n^{(2)} = (n+1)\frac{\mu_n}{\omega_n^2}(\cos\omega_n t-1)+\frac{2\omega_l\varpi_n-(n+1)\,\beta_n}{\omega_n^2-(2\omega_l)^2}(\cos 2\omega_l t-\cos\omega_n t), \tag{4.7}$$

which in turn yields

$$\phi_n^{(2)} = \frac{\varpi_n}{n+1}\sin 2\omega_l t+\frac{\mu_n}{\omega_n}\sin\omega_n t+\frac{2\omega_l\varpi_n-(n+1)\,\beta_n}{\omega_n^2-(2\omega_l)^2}(2\omega_l\sin 2\omega_l t-\omega_n\sin\omega_n t). \tag{4.8}$$

If $2\omega_l$ is not close to ω_n, $\phi^{(2)}$ and $\eta^{(2)}$ are oscillatory and are of order one, so that

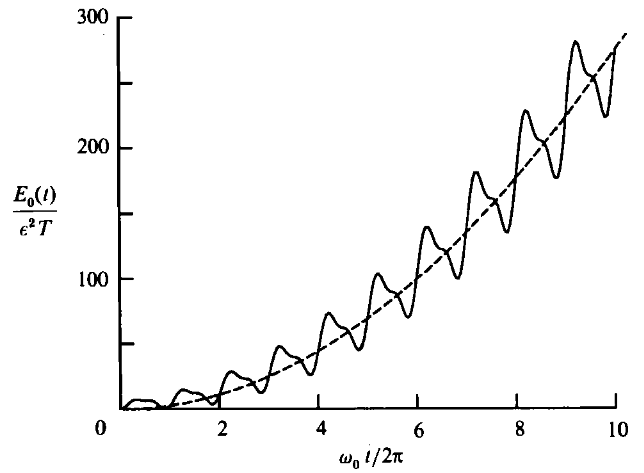

FIGURE 1. The energy level in the induced pulsation mode according to the direct perturbation scheme, with $l = 6$ and the dashed curve indicating the t^2 growth.

the second-order solutions can be neglected compared with the first-order quantities. The second-order terms may become important if $2\omega_l \rightarrow \omega_n$. Since we are mainly concerned with the excitation of volume pulsations, we may let $n = 0$ and $2\omega_l \rightarrow \omega_0$, in which case the solutions (4.7) and (4.8) represent the excited volumetric mode in the form

$$\left.\begin{aligned} \eta_0^{(2)} &= \frac{\mu_0}{\omega_0^2}(\cos\omega_0 t - 1) + \frac{\omega_0\varpi_0 - \beta_0}{2\omega_0^2}\omega_0 t\sin\omega_0 t, \\ \phi_0^{(2)} &= \frac{\omega_0\varpi_0 + \beta_0 + 2\mu_0}{2\omega_0}\sin\omega_0 t - \frac{\varpi_0}{2\omega_0}\omega_0 t\cos\omega_0 t. \end{aligned}\right\} \tag{4.9}$$

This result shows that the excited volumetric mode is oscillatory but its amplitude grows linearly with time. This is not an acceptable solution; it violates the assumption that $|\eta| < 1$, which is essential to ensure the convergence of the Taylor expansion in the boundary condition (2.10). According to Prosperetti & Lu (1988), large-amplitude cavitations are unlikely to occur for bubbles in the upper ocean surface layer, which justifies the assumption $|\eta| < 1$. In any event, the geometrical constraint on the bubble oscillation requires that its amplitude must be smaller than the radius of the bubble at equilibrium. The ever-growing-amplitude solution simply indicates the breakdown of the direct perturbation scheme at large time.

The excitation of the volumetric mode (4.9) implies of course that the energy in the surface distortion mode (the lth mode) is transferred to the pulsation mode. From (4.9), it is easy to calculate the energy contained in the volumetric mode according to the definition (3.7):

$$\begin{aligned} E_0(t) = \epsilon^2 \frac{T(l-1)(l+2)}{256(l+1)(2l+1)^2}\{&(4l-1)^2(\omega_0 t)^2 + 8(8l+5)^2(1-\cos\omega_0 t) + (464l^2 \\ &+ 504l + 125)\sin^2\omega_0 t + (4l-1)[4(8l+5) + 2(12l+7)\cos\omega_0 t]\,\omega_0 t\sin\omega_0 t\}, \end{aligned} \tag{4.10}$$

where (4.4) has been utilized to calculate ϖ_0, β_0 and μ_0. This pulsation energy increases in proportion to the square of t, as shown in figure 1. This eventually does not conform with the principle of energy conservation (3.6) that the bubble energy cannot exceed its initial value. As time increases, the ever-growing energy level

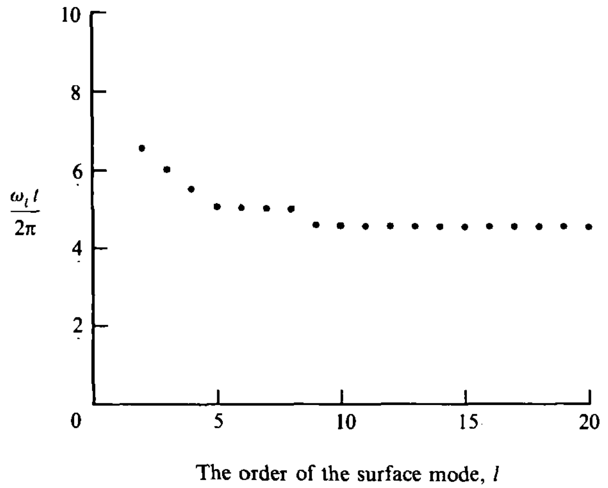

FIGURE 2. The time when the pulsation energy reaches the initial bubble energy for $\epsilon = 0.1$.

exceeds the initial bubble energy, indicating the failure of the perturbation theory. The time at which the energy in the pulsation mode reaches the initial bubble energy can be found by solving the equation

$$(4l-1)^2(\omega_0 t)^2 + 8(8l+5)^2(1-\cos\omega_0 t) + (4l-1)[4(8l+5) + 2(12l+7)\cos\omega_0 t] \\ \times \omega_0 t \sin\omega_0 t + (464l^2 + 504l + 125)\sin^2\omega_0 t = \frac{256(l+1)(2l+1)}{\epsilon^2}, \quad (4.11)$$

for t, which results from equating (4.10) to the initial bubble energy. This time sets a maximum time within which the direct perturbation scheme could conceivably be relevant. As an example, the solutions of (4.11) are plotted in figure 2 for $\epsilon = 0.1$, which is likely to be the order of surface distortion for bubbles in the ocean surface layer and corresponds to the case examined by Longuet-Higgins (1989*b*). In this case, the maximum time depends on the order l of the surface distortion mode but the variation for different l at l larger than 5 is very small. For the problem of oceanic noise generation, the frequency range is from about 1 kHz of tens of kHz (Wenz 1962; Urick 1967) and the bubble radius from 0.01 cm to 1 cm (Minnaert 1933; Fitzpatrick & Strasberg 1957; Toba 1961). In this case, the resonance condition $\omega_0 = 2\omega_l$ requires that l should be well above 5. Thus, it can be seen from figure 2 that the maximum time for which the direct perturbation model might possibly be relevant is only about 4 to 5 cycles of the bubble oscillation even when the dissipation is neglected. The method is fundamentally inadequate for relating the strength of a periodic volumetric oscillation to its origin in the bubble distortion.

5. Multiple-scales formulation

In anticipating that the modes which nonlinearly interact with each other will be modulated, we introduce a slow timescale to characterize this modulation $\tau = \epsilon t$, and regard the dependent variables η and ϕ as functions of both t and τ. This technique of introducing two timescales in the study of bubble oscillations has been used by

Hall & Seminara (1980). In this formulation, the time derivatives in the boundary conditions (2.10) are replaced by

$$\frac{\partial}{\partial t}+\epsilon\frac{\partial}{\partial \tau}.$$

Substituting the expansions (4.1) into the governing equation (2.1) reveals that both the first- and second-order velocity potential satisfies the Laplace equation. Applying the expansion to the boundary condition (2.10) and grouping terms of the same order in ϵ, it can be shown that the first-order boundary condition is still linear and homogeneous as in the previous section, but the second-order one is supplemented by both the nonlinear terms determined from the first-order solutions and terms involving derivatives with respect to τ, which are found to be

$$\left.\begin{aligned}\frac{\partial\eta^{(2)}}{\partial t}-\frac{\partial\phi^{(2)}}{\partial r} &= \eta^{(1)}\frac{\partial^2\phi^{(1)}}{\partial r^2}-\nabla_s\eta^{(1)}\cdot\nabla_s\phi^{(1)}-\frac{\partial\eta^{(1)}}{\partial\tau},\\ \frac{\partial\phi^{(2)}}{\partial t}+T(2+\nabla_s^2)\,\eta^{(2)}-3\gamma(1+2T)\,\overline{\eta^{(2)}} &= 2T[(\eta^{(1)})^2+\eta^{(1)}\nabla_s^2\eta^{(1)}]-\eta^{(1)}\frac{\partial^2\phi^{(1)}}{\partial r\,\partial t}\\ &\quad-\tfrac{1}{2}(\nabla\phi^{(1)})^2+3\gamma(1+2T)\,[\overline{(\eta^{(1)})^2}\\ &\quad-\tfrac{3}{2}(1+\gamma)\,\overline{\eta^{(1)}}^2]-\frac{\partial\phi^{(1)}}{\partial\tau}.\end{aligned}\right\}\quad(5.1)$$

The initial condition (2.11) now becomes

$$\left.\begin{aligned}&\eta^{(1)}=S_l(\theta,\varphi)\quad\text{and}\quad\frac{\partial\eta^{(1)}}{\partial t}=0\\ &\eta^{(2)}=\frac{\partial\eta^{(2)}}{\partial t}=0\end{aligned}\right\}\quad\text{at}\quad t=\tau=0.\qquad(5.2)$$

Supposing that the first-order problem has solutions of the form

$$\left.\begin{aligned}\eta^{(1)}&=\sum_n\eta_n^{(1)}(t,\tau)\,S_n(\theta,\varphi),\\ \phi^{(1)}&=\sum_n\phi_n^{(1)}(t,\tau)\,S_n(\theta,\varphi)\frac{1}{r^{n+1}},\end{aligned}\right\}\qquad(5.3)$$

the linear boundary conditions then give the equation for $\eta_n^{(1)}(t,\tau)$:

$$\frac{\mathrm{d}^2\eta_n^{(1)}}{\mathrm{d}t^2}+\omega_n^2\,\eta_n^{(1)}=0,$$

which has the general solution

$$\eta_n^{(1)}=\tfrac{1}{2}[C_n\,\mathrm{e}^{\mathrm{i}\omega_n t}+C_n^*\,\mathrm{e}^{-\mathrm{i}\omega_n t}],\qquad(5.4)$$

where complex notations are used so that C_n is the complex amplitude of the nth mode with C_n^* being its complex conjugate. To account for the interactions between the initial surface distortion (the lth mode) and the volumetric mode, we assume that the complex amplitude C_n for these two modes are functions of the slow time τ, so that

$$C_n=\begin{cases}C_n(\tau) & \text{when } n=0,l\\ \text{constant} & \text{otherwise.}\end{cases}$$

From this, it can be shown that $C_n=0$ unless $n=0$ or $n=l$ because both $\eta_n^{(1)}$ and its derivative with respect to t vanish at the initial instant for $n \neq 0$ and $n \neq l$. Thus the series solutions (5.3) degenerates to

$$\left.\begin{aligned} \eta^{(1)} &= \eta_0^{(1)} + \eta_l^{(1)} S_l(\theta,\varphi), \\ \phi^{(1)} &= \phi_0^{(1)} \frac{1}{r} + \phi_l^{(1)} S_l(\theta,\varphi) \frac{1}{r^{l+1}}, \end{aligned}\right\} \tag{5.5}$$

where $\eta_0^{(1)}$ and $\eta_l^{(1)}$ are given by (5.4) and $\phi_0^{(1)}$ and $\phi_l^{(1)}$ are defined, with $n = 0$ and $n = l$, by

$$\phi_n^{(1)} = -\frac{\mathrm{i}\omega_n}{2(n+1)} [C_n(\tau)\, \mathrm{e}^{\mathrm{i}\omega_n t} - C_n^*(\tau)\, \mathrm{e}^{-\mathrm{i}\omega_n t}]. \tag{5.6}$$

The complex amplitudes C_0 and C_l are to be determined from the next-order problem with the initial conditions

$$C_0 = 0 \quad \text{and} \quad C_l = 1 \quad \text{at} \quad \tau = 0, \tag{5.7}$$

which follows from substituting (5.4) and (5.5) into the initial conditions (5.2).

With $\eta^{(1)}$ and $\phi^{(1)}$ given by (5.4) and (5.5), the terms on the right-hand sides of (5.1) can be evaluated. Furthermore, $\eta^{(2)}$ and $\phi^{(2)}$ can be assumed to be of the form (4.6), so that two equations for $\eta_0^{(2)}$ and $\eta_l^{(2)}$ can be derived by respectively multiplying (5.1) by S_0 and S_l, and then taking averages of the results. When this is done, we find that

$$\frac{\mathrm{d}^2\eta_0^{(2)}}{\mathrm{d}t^2} + \omega_0^2\, \eta_0^{(2)} = \frac{1}{8(2l+1)} \left[\omega_l^2 \frac{4l+9}{l+1} - 2\omega_0^2\right] C_l^2\, \mathrm{e}^{2\mathrm{i}\omega_l t} - \mathrm{i}\omega_0 \frac{\partial C_0}{\partial \tau} \mathrm{e}^{\mathrm{i}\omega_0 t} + Q_0; \tag{5.8}$$

$$\frac{\mathrm{d}^2\eta_l^{(2)}}{\mathrm{d}t^2} + \omega_l^2\, \eta_l^{(2)} = \tfrac{1}{4}[3\omega_l^2 - (l-1)\,\omega_0^2 - 3\omega_0\,\omega_l]\, C_0\, C_l^*\, \mathrm{e}^{\mathrm{i}(\omega_0-\omega_l)t} - \mathrm{i}\omega_l \frac{\partial C_l}{\partial \tau} \mathrm{e}^{\mathrm{i}\omega_l t} + Q_l, \tag{5.9}$$

where Q_0 and Q_l are introduced to denote

$$\begin{aligned} 4Q_0 &= [\tfrac{3}{2}\omega_0^2(\gamma+2) + T(3\gamma-1)]\, C_0^2\, \mathrm{e}^{2\mathrm{i}\omega_0 t} \\ &\quad + [\tfrac{3}{2}\omega_0^2\gamma + T(3\gamma-1)]\, C_0\, C_0^* + \frac{1}{2(2l+1)}\left[\frac{3\omega_l^2}{l+1} - 2\omega_0^2\right] C_l\, C_l^*; \\ 4Q_l &= [3\omega_l^2 - (l-1)\,\omega_0^2 + 3\omega_0\,\omega_l]\, C_0\, C_l\, \mathrm{e}^{\mathrm{i}(\omega_0+\omega_l)t} \\ &\quad + \frac{2l+1}{2}\left[T(l+1)\,(l^3+12l^2+9l-18)\,\overline{S_l^3} - \frac{5\omega_l^2}{l+1}\,\overline{(\boldsymbol{\nabla}_\mathrm{s} S_l)^2 S_l}\right] C_l^2\, \mathrm{e}^{2\mathrm{i}\omega_l t} \\ &\quad + \frac{2l+1}{2}\left[T(l+1)\,(3l^3+16l^2+7l-22)\,\overline{S_l^3} - \frac{\omega_l^2}{l+1}\,\overline{(\boldsymbol{\nabla}_\mathrm{s} S_l)^2 S_l}\right] C_l\, C_l^*, \end{aligned}$$

and we have omitted complex-conjugate terms on the right-hand sides of (5.8) and (5.9) to save space.

Now, the usual argument of the multiple-scales technique applies. To render the series expansion (4.1) uniformly valid at large time, it is required that the higher-order terms in the expansion should not be more singular than the lower-order terms (Nayfeh & Mook 1979). The behaviour of $\eta_0^{(2)}$ and $\eta_l^{(2)}$ depends on the forcing terms on the right-hand sides of (5.8) and (5.9), of which those in proportion to $\exp(\pm\mathrm{i}\omega_0 t)$ in (5.8) and to $\exp(\pm\mathrm{i}\omega_l t)$ in (5.9) are secular terms that would result in the ever-growing, and hence unacceptable, solution. These solutions have worse behaviour at large time than the first-order solution (5.5) so that must be annihilated. This can be done by letting the secular forcing terms in (5.8) and (5.9) vanish, which results in two equations to determine the complex amplitudes $C_0(\tau)$ and $C_l(\tau)$. Depending on whether ω_0 is away from, close to or exactly equal to $2\omega_l$, the secular forcing terms are different for each case. The simplest case is when $2\omega_l$ is not close to ω_0, which is the trivial case of non-resonant oscillations. In this event, the only secular terms in (5.8) and (5.9) are those involving derivatives with respect to τ. Setting them to zero, it follows immediately that

$$\frac{\mathrm{d}C_0}{\mathrm{d}\tau} = 0, \quad \frac{\mathrm{d}C_l}{\mathrm{d}\tau} = 0.$$

The oscillations are not modulated, as is expected because there is no interaction between the two modes in this non-resonance case. The amplitudes are then identical to the initial value (5.7). For the cases where ω_0 is exactly equal to or close to $2\omega_l$, the solutions are much more complicated, and these are examined in the following sections.

6. Exact resonance $\omega_0 = 2\omega_l$

In the case of $\omega_0 = 2\omega_l$, exact resonance occurs between the lth mode and the volumetric mode. From (5.8) and (5.9), it is clear that the secular forcing terms in this case are the first two terms on the right-hand sides. By setting these terms to zero, the equations for determining the complex amplitudes C_0 and C_l can be found to be

$$\left.\begin{aligned} \frac{\mathrm{d}C_0}{\mathrm{d}\tau} - \mathrm{i}\frac{(4l-1)\,\omega_l}{16(2l+1)\,(l+1)} C_l^2 &= 0, \\ \frac{\mathrm{d}C_l}{\mathrm{d}\tau} - \mathrm{i}\frac{(4l-1)\,\omega_l}{4} C_0\, C_l^* &= 0. \end{aligned}\right\} \tag{6.1}$$

It can be recognized that these equations are similar to those associated with oscillation systems of two degrees of freedom with quadratic nonlinearities and their solutions can be found by following the procedure of, say, Nayfeh & Mook (1979), which we give briefly in the following. It is not surprising that, despite the complicated physical background, the amplitudes of the bubble oscillations conform with the simple equations (6.1); the resonant interactions between the two modes resemble precisely an oscillation system of two degrees of freedom and the truncation of the Taylor expansion (2.9) up to the order η^2 results in the quadratic nonlinear behaviour.

To facilitate the solution, we write the complex amplitudes C_0 and C_l in terms of their amplitude and phase

$$C_0 = a_0\,\mathrm{e}^{\mathrm{i}\alpha_0}, \quad C_l = a_l\,\mathrm{e}^{\mathrm{i}\alpha_l}, \tag{6.2}$$

which separates the modulation of the bubble oscillation into an amplitude and a phase component, where a_0, a_l, α_0 and α_l are all real functions of τ. On substituting (6.2) into (6.1), the real parts of these equations lead to

$$\frac{\mathrm{d}a_0}{\mathrm{d}\tau} + \frac{(4l-1)\,\omega_l}{16(2l+1)\,(l+1)} a_l^2 \sin\Theta = 0, \tag{6.3a}$$

$$\frac{\mathrm{d}a_l}{\mathrm{d}\tau} - \frac{(4l-1)\,\omega_l}{4} a_0\, a_l \sin\Theta = 0, \tag{6.3b}$$

where for brevity Θ replaces $2\alpha_l-\alpha_0$. Eliminating $\sin\Theta$ from these equations, it is found that

$$\frac{\mathrm{d}a_0^2}{\mathrm{d}\tau}+\frac{1}{4(2l+1)(l+1)}\frac{\mathrm{d}a_l^2}{\mathrm{d}\tau}=0. \tag{6.4}$$

This is in fact the energy equation for the oscillations, which can be shown by substituting the solution (5.5), with $\eta_0^{(1)}$, $\eta_l^{(1)}$, $\phi_0^{(1)}$ and $\phi_l^{(1)}$ given by (5.4) and (5.6), into the energy equation (3.6). This immediately gives (3.6) in the form

$$C_0C_0^*+\frac{1}{4(2l+1)(l+1)}C_lC_l^*=\frac{1}{4(2l+1)(l+1)}, \tag{6.5}$$

which, by taking the derivative with respect to τ, is precisely identical to (6.4).

The imaginary parts of (6.1) yield two equations for α_0 and α_l:

$$a_0\frac{\mathrm{d}\alpha_0}{\mathrm{d}\tau}-\frac{(4l-1)\,\omega_l}{16(2l+1)(l+1)}a_l^2\cos\Theta=0, \tag{6.6a}$$

$$a_l\frac{\mathrm{d}\alpha_l}{\mathrm{d}\tau}-\frac{(4l-1)\,\omega_l}{4}a_0a_l\cos\Theta=0, \tag{6.6b}$$

which can be further combined to give an equation for $\Theta=2\alpha_l-\alpha_0$:

$$a_0\frac{\mathrm{d}\Theta}{\mathrm{d}\tau}+\frac{(4l-1)\,\omega_l}{4}\left[\frac{1}{4(2l+1)(l+1)}a_l^2-2a_0^2\right]\cos\Theta=0.$$

By using the chain rule of differentiation to write

$$\frac{\mathrm{d}\Theta}{\mathrm{d}\tau}=\frac{\mathrm{d}\Theta}{\mathrm{d}a_0}\frac{\mathrm{d}a_0}{\mathrm{d}\tau}, \tag{6.7}$$

and utilizing (6.3a) to eliminate $\mathrm{d}a_0/\mathrm{d}\tau$ with a_l given in terms of a_0 by (6.5), this equation can be integrated directly with the result

$$a_0\cos\Theta\left[\frac{1}{4(2l+1)(l+1)}-a_0^2\right]=\text{constant}.$$

Since $a_0=0$ at $\tau=0$, the integration constant must be set to zero, which in turn gives $\cos\Theta=0$. Thus, (6.6) reduces to

$$\frac{\mathrm{d}\alpha_0}{\mathrm{d}\tau}=\frac{\mathrm{d}\alpha_l}{\mathrm{d}\tau}=0,$$

which means that the oscillations do not undergo phase modulation. From the initial conditions (5.2) and the fact that the amplitude of the volumetric mode increases at small τ ($\mathrm{d}a_0/\mathrm{d}\tau>0$), it follows that $\alpha_l=0$ and $\alpha_0=\frac{1}{2}\pi$. This yields $\sin\Theta=-1$ and the amplitude equation (6.3) for a_0 becomes

$$\frac{\mathrm{d}a_0}{\mathrm{d}\tau}-\frac{(4l-1)\,\omega_l}{4}\left[\frac{1}{4(2l+1)(l+1)}-a_0^2\right]=0,$$

where (6.5) has been used to eliminate a_l. This equation can be integrated immediately to give

$$a_0(\tau)=\frac{1}{2[(2l+1)(l+1)]^{\frac{1}{2}}}\tanh\left[\frac{(4l-1)\,\omega_0\,\tau}{16[(2l+1)(l+1)]^{\frac{1}{2}}}\right]. \tag{6.8}$$

The amplitude a_l can then be found from this and (6.5). Collecting all the results

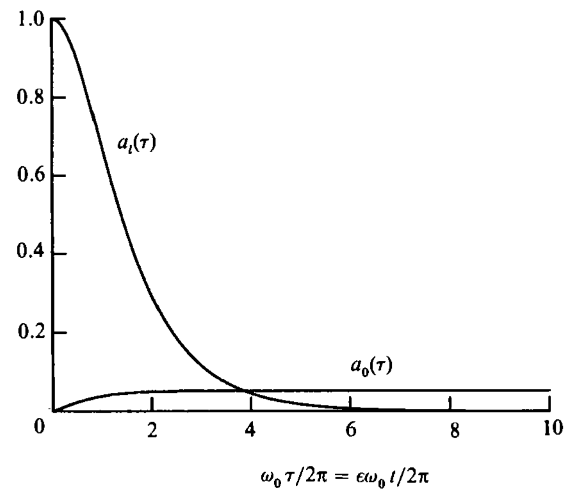

FIGURE 3. Amplitude modulations in the case of exact resonance with $l = 6$.

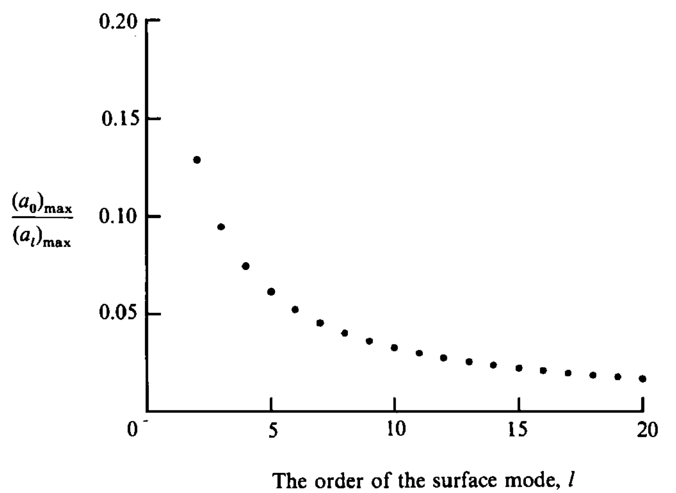

FIGURE 4. The ratio of the maximum amplitudes of the two modes in exact resonance.

derived above, we see that in this case of exact resonance, the oscillations only experience modulations and the modulations are monotonic functions of τ; the surface distortion mode continuously feeds energy into the pulsation mode until the initial bubble energy is all in the volumetric mode. These results are all plotted in figure 3.

It is interesting to note from figure 3 that the maximum amplitude of the induced volumetric mode is much smaller than that of the surface distortion mode. This is because pulsations need more energy than pure shape distortions in order to maintain the same perturbation amplitude. In the present problem, the finite initial bubble energy is given in the form of pure shape oscillations, so that, when this finite amount of energy is converted into the volumetric mode to energize pulsations, the amplitude of the pulsations must be smaller than that of the surface mode with the same energy level. The maximum amplitude of the surface distortion can be found by simply letting a_0 be zero in the energy equation (6.5) and that of the pulsation mode can be obtained by letting $\tau \to +\infty$ in (6.8), which leads to

$$(a_l)_{\max} = 1, \quad (a_0)_{\max} = \frac{1}{2[(2l+1)(l+1)]^{\frac{1}{2}}}.$$

The ratio of the two is shown in figure 4 as a function of the order of the surface mode l, which can be seen to decrease, from about 0.13 at $l = 2$, continuously to zero as l increases. The decrease at large l is in proportion to $1/l$. This result is in contrast to the direct perturbation prediction that large-amplitude pulsation is possible at resonance. It is clear that the principle of energy conservation leads to the conclusion that only small-amplitude pulsations can be excited. This becomes even more pronounced in the case of near resonance, which we examine in the next section.

7. The case of near resonance

In the case of near resonance where ω_0 is close, but not exactly equal, to $2\omega_l$, the interactions between the two modes show different characteristics. To demonstrate this, we assume that

$$\omega_0 = 2\omega_l(1+\epsilon).$$

From this and since $\tau = \epsilon t$, the vanishing of the secular forcing terms in (5.8) and (5.9) yields

$$\left.\begin{aligned} \frac{dC_0}{d\tau} - i\frac{(4l-1)\,\omega_l}{16(2l+1)(l+1)} C_l^2\, e^{-i\omega_0\tau} &= 0, \\ \frac{dC_l}{d\tau} - i\frac{(4l-1)}{4} C_0 C_l^*\, e^{i\omega_0\tau} &= 0. \end{aligned}\right\} \tag{7.1}$$

Comparing this with (6.1) for the case of exact resonance, it is clear that the difference is the presence of $\exp(\pm\omega_0\tau)$ in (7.1). Following the same procedure, we introduce (6.2) into (7.1), the real and imaginary parts of which can be shown to give four equations identical to (6.3) and (6.6), provided that we now define Θ as

$$\Theta = 2\alpha_l - \alpha_0 - \omega_0\tau.$$

The energy equation (6.4) can then be derived in the same way and the equation for Θ now assumes the form

$$a_0\frac{d\Theta}{d\tau} + \frac{(4l-1)\,\omega_l}{4}\left[\frac{1}{4(2l+1)(l+1)}a_l^2 - 2a_0^2\right]\cos\Theta + a_0\omega_0 = 0.$$

By making use of (6.7), we can change $d\tau$ to da_0, which leads to

$$a_l^2\cos\Theta\, da_0 - a_l^2 a_0\sin\Theta d\Theta - 8(2l+1)(l+1)a_0^2\cos\Theta\, da_0 + \frac{32(2l+1)(l+1)}{(4l-1)}a_0\, da_0 = 0.$$

The first three terms can be shown to be equal to $d(a_l^2 a_0\cos\Theta)$. The integration of the above equation then gives

$$a_l^2\cos\Theta + \frac{16(2l+1)(l+1)}{(4l+1)}a_0 = 0, \tag{7.2}$$

where the integration constant has been set to zero according to the initial condition (5.2). From this result, the amplitude equation for a_0 can be found as

$$\frac{da_0}{d\tau} = \frac{(4l-1)\,\omega_l}{16(2l+1)(l+1)}\left[a_l^4 - \frac{256(2l+1)^2(l+1)^2}{(4l-1)^2}a_0^2\right]^{\frac{1}{2}}. \tag{7.3}$$

Using (6.5) to eliminate a_l, the integration of this equation can be transformed to the form

$$\int_0^{a_0/\lambda_-} \frac{1}{[(1-y^2)(1-k^2y^2)]^{\frac{1}{2}}}\,\mathrm{d}y = \lambda_+ \frac{4l-1}{4}\omega_l \tau, \tag{7.4}$$

where $k = \lambda_-/\lambda_+$ with $\lambda_\pm$ defined by

$$\lambda_\pm = \frac{2}{(4l-1)}\left[\left(1+\frac{(4l-1)^2}{16(2l+1)(l+1)}\right)^{\frac{1}{2}} \pm 1\right].$$

The equation (7.4) is a standard form for the elliptic functions, which has the solution

$$a_0(\tau) = \lambda_- \left|\mathrm{sn}\left(\lambda_+ \frac{4l-1}{4}\omega_l \tau, k\right)\right|, \tag{7.5}$$

where sn is the Jacobian elliptic function (Gradshteyn & Ryzhik 1980).

With a_0 given by (7.5), a_l can be found from the energy equation (6.5), and these are both plotted in figure 5. It can be seen that the amplitude modulations are oscillatory in time, indicating that energy is cyclically exchanged between one mode and the other. Again, both modes are bounded and the sum of their energies is always equal to the initial bubble energy.

From the results (7.2) and (6.6), the equation for the phase modulation α_0 can be obtained:

$$\frac{\mathrm{d}\alpha_0}{\mathrm{d}\tau} + \tfrac{1}{2}\omega_0 = 0,$$

which, with the initial condition $\alpha_0(0) = \frac{1}{2}\pi$, immediately yields

$$\alpha_0(\tau) = \tfrac{1}{2}\pi + \tfrac{1}{2}\omega_0 \tau.$$

On substituting this result, together with (6.2), into (5.4), the complete solution for the volumetric mode becomes

$$\eta_0^{(1)} = a_0(\tau)\sin[\omega_0 t(1+0.5\epsilon)].$$

It is clear that the volume pulsation is not exactly at the frequency for the linear problem; nonlinear interactions cause a modulation that changes the pulsation frequency. The modulation of frequency due to nonlinear interactions has previously been observed and studied for oscillations of liquid drops (e.g. Trinh & Wang 1982; Tsamopoulos & Brown 1983). The surface distortion mode also experiences a phase modulation which can be found from (7.2) and (6.6*b*). It can be shown that α_l is given by

$$\frac{\mathrm{d}\alpha_l}{\mathrm{d}\tau} + 2(2l+1)(l+1)\,\omega_0 \frac{a_0^2}{a_l^2} = 0,$$

which can be integrated, with $\alpha_l(0) = 0$, to give

$$\alpha_l(\tau) = -2(2l+1)(l+1)\,\omega_0 \int_0^\tau \frac{a_0^2(\tau')}{a_l^2(\tau')}\,\mathrm{d}\tau'.$$

With a_0 and a_l given by (7.5) and (6.5), this can be readily evaluated. The phase modulation for the surface mode also causes a reduction in the oscillation frequency, because α_l is negative.

As in the case of exact resonance, figure 5 shows that the volumetric pulsation is much smaller in amplitude than the surface distortion. The maximum amplitude for the surface mode is 1, which is clear both from figure 5 and from the energy equation

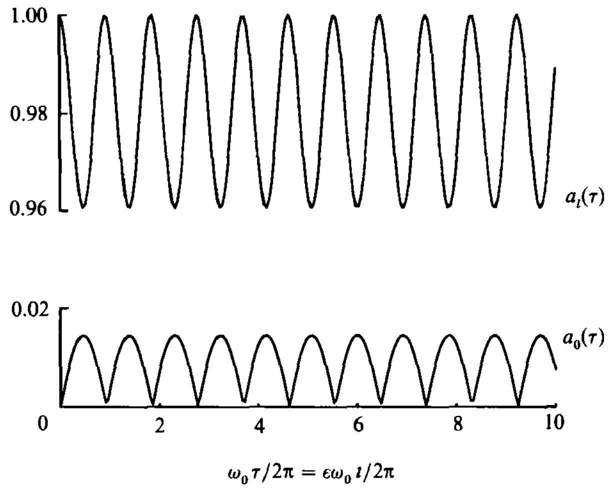

FIGURE 5. Amplitude modulations in the case of near resonance with $l = 6$.

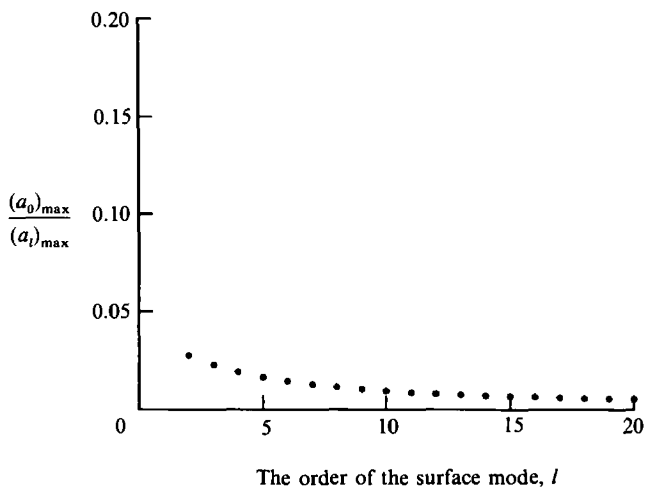

FIGURE 6. The ratio of the maximum amplitudes of the two modes in the near resonance.

(6.5) with $a_0 = 0$, and the maximum pulsation amplitude can be found by setting $\mathrm{d}a_0/\mathrm{d}\tau$ to zero in (7.3), which shows that this maximum is simply λ_-. Thus, the ratio of the two can be written as

$$\frac{(a_0)_{\max}}{(a_l)_{\max}} = \frac{2}{4l-1}\left[\left(1+\frac{(4l-1)^2}{16(2l+1)(l+1)}\right)^{\frac{1}{2}}-1\right].$$

For $l = 2$ this is about 0.03 and for large l it decreases in proportion to $1/l$, as in the case of exact resonance, which is clearly shown in figure 6. Comparing figure 6 with figure 4, it can be seen that the volume pulsation has an even smaller amplitude in the case of near resonance, because some of the initial bubble energy in this case always remains in the surface mode, which is evident from figure 5 where we can see that the surface mode never reaches zero amplitude.

8. Discussion and conclusion

We have examined the problem of bubble pulsations due to the resonant nonlinear interactions of bubble shape oscillations. The nonlinear coupling between the surface mode and the volumetric mode is taken into account by resorting to the technique of multiple scales, which formulates the problem as an oscillation system with two degrees of freedom. The results show that, when resonance occurs, both the surface mode and the induced pulsations experience modulation. It is this modulation that causes energy conversion from one mode to the other. We have also shown that a direct perturbation approach to this problem fails to conform with the principle of energy conservation because the long-term coupling is not taken into proper account.

At exact resonance, the energy conversion is monotonically from the surface mode to the volumetric mode, but energy is exchanged cyclically between the two near, but not at, resonance, indicating both amplitude and phase modulation. It has been shown that both the surface distortion and the volume pulsation are bounded and the sum of energies in the two interacting modes is always equal to the initial bubble energy; an increase of amplitude in one mode is always accompanied by a decrease in the other. Because volume pulsation contains much more energy than shape distortion if the two maintain the same amplitude of oscillation, it is shown that the pulsations induced by a pure initial surface distortion have a much smaller amplitude than the surface mode with the same energy level. If the interacting surface mode is of high order, the induced pulsation has an increasingly smaller amplitude as the order of the surface distortion increases.

The analysis presented in this paper gives an upper bound on the volume pulsation because it neglects both radiation and dissipation effects. These are important features of bubble oscillations that can also influence the conversion of energy from the surface mode to the volumetric mode. However, they hardly increase the energy in the volumetrical mode. The volume pulsations due to nonlinear interactions of surface modes are very small; damping effects would dissipate part of the initial bubble energy so that the energy converted to the volumetric mode would be even less if these effects were considered.

It should be pointed out that, though our analysis shows that an isolated bubble with (only) initial distortion is unlikely to radiate appreciable sound by nonlinear interactions, the book on this issue can not be closed; there may be cases where this nonlinear interaction mechanism plays an important role in the energy conversion process. Bubble–bubble interaction is a possible example. Suppose that a bubble oscillates at 10 kHz. Its energy is likely to be radiated and dissipated in about 10 cycles, that is, the acoustically active time of this bubble is about 10^{-3} s. In the ocean environment, bubbles usually move, owing to buoyancy or background pressure gradients, with a typical velocity of the order 0.2 m/s. Thus a bubble may cover a distance of the order 2×10^{-4} m during its active lifetime. If the bubble is in a bubble cloud with the interbubble distance less than this value (which means a gas concentration larger than 1% by volume), it is possible that new energy is gained by the bubble distortion mode, owing to bubble–bubble interaction, before its initial energy is dissipated. Thus part of the energy involved in the bubble–bubble interaction may then sustain the surface mode to excite the volumetrical mode long enough to be significant.

Another possible case concerns the energy transfer from a turbulent bubbly flow to sound. In this case, the background turbulent flow has a characteristic lengthscale at the bubble oscillation frequencies that is small in comparison with the bubble size.

Thus, as analysed by Crighton & Ffowcs Williams (1969), the turbulent flow cannot directly ring the bubble because its pressure fluctuations are not coherent over the bubble surface. However, this spacially non-uniform turbulent pressure on the bubble wall may deform the bubble, exciting bubble surface modes. These modes may then transfer their energy into sound by nonlinear interactions. Because the background turbulent flow is very energetic, it is possible that the sound from this 'cascade' energy transfer process is an appreciable component of the bubble-related sound.

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Appendix

In this Appendix, we first prove the following identities which are used to derive the energy equation (3.2)

$$\frac{\mathrm{d}E_\mathrm{K}}{\mathrm{d}t} = \int \frac{\mathrm{D}}{\mathrm{D}t}[\tfrac{1}{2}(\nabla\phi)^2]\,\mathrm{d}^3\boldsymbol{x}, \tag{A 1}$$

$$\frac{\mathrm{d}E_\mathrm{W}}{\mathrm{d}t} = \int_{S_\infty} \nabla\phi\cdot\boldsymbol{n}_\infty\,\mathrm{d}s_\infty, \tag{A 2}$$

$$\frac{\mathrm{d}E_\mathrm{B}}{\mathrm{d}t} = -\int_{S_\mathrm{B}} p_\mathrm{B}(\nabla\phi\cdot\boldsymbol{n}_\mathrm{B})\,\mathrm{d}s_\mathrm{B}, \tag{A 3}$$

$$\frac{\mathrm{d}E_T}{\mathrm{d}t} = T\int_{S_\mathrm{B}} \nabla\cdot\boldsymbol{n}_\mathrm{B}(\nabla\phi\cdot\boldsymbol{n}_\mathrm{B})\,\mathrm{d}s_\mathrm{B}, \tag{A 4}$$

where E_K, E_W, E_B and E_T are respectively the total kinetic energy in water, the work done at infinity by p_∞, the internal energy of the bubble and the energy associated with surface tension, and are given respectively by

$$E_\mathrm{K} = \int \tfrac{1}{2}(\nabla\phi)^2\,\mathrm{d}^3\boldsymbol{x}, \tag{A 5}$$

$$E_\mathrm{W} = \int_0^t\int_{S_\infty} \nabla\phi\cdot\boldsymbol{n}_\infty\,\mathrm{d}s_\infty\,\mathrm{d}t, \tag{A 6}$$

$$E_\mathrm{B} = \frac{V_\mathrm{B}\,p_\mathrm{B}}{\gamma-1}, \tag{A 7}$$

$$E_T = T\int_{S_\mathrm{B}} \mathrm{d}s_\mathrm{B}, \tag{A 8}$$

where (A 5), (A 6) and (A 8) are obvious and (A 7) follows from the definition that the internal energy per unit volume for isotropic gas is equal to the pressure in the gas divided by $\gamma-1$ (see, for example, Landau & Lifshitz 1959) and noticing that the pressure in the bubble is assumed to be uniform.

The identity (A 1) becomes obvious from the transport theorem

$$\int\frac{\mathrm{D}F}{\mathrm{D}t}\,\mathrm{d}^3\boldsymbol{x} = \frac{\mathrm{d}}{\mathrm{d}t}\int F\,\mathrm{d}^3\boldsymbol{x} - \int F\nabla^2\phi\,\mathrm{d}^3\boldsymbol{x}, \tag{A 9}$$

with $F = \frac{1}{2}(\nabla\phi)^2$, the second term on the right-hand side being identically zero from (2.1). By taking the derivative of (A 6) with respect to t, it is also obvious that (A 2) holds. To demonstrate (A 3), we note that the t-derivative of (A 7) and the adiabatic law (2.6) lead to

$$\frac{\mathrm{d}E_{\mathrm{B}}}{\mathrm{d}t} = -p_{\mathrm{B}}\frac{\mathrm{d}V_{\mathrm{B}}}{\mathrm{d}t}. \tag{A 10}$$

From the definition $\boldsymbol{n}_{\mathrm{B}} = \nabla f/|\nabla f|$ and the kinematic boundary condition (2.2), we have

$$\nabla\phi\cdot\boldsymbol{n}_{\mathrm{B}} = \frac{\partial\eta}{\partial t}\frac{1}{|\nabla f|}, \tag{A 11}$$

so that, on integrating over the bubble surface,

$$\int_{S_{\mathrm{B}}}\nabla\phi\cdot\boldsymbol{n}_{\mathrm{B}}\,\mathrm{d}s_{\mathrm{B}} = \int_{S_{\mathrm{B}}}\frac{\partial\eta}{\partial t}\frac{1}{|\nabla f|}\mathrm{d}s_{\mathrm{B}} = \frac{\mathrm{d}V_{\mathrm{B}}}{\mathrm{d}t}, \tag{A 12}$$

where we have made use of the formula

$$\int_{S_{\mathrm{B}}}F\frac{1}{|\nabla f|}\mathrm{d}s_{\mathrm{B}} = \int_0^{\pi}\int_0^{2\pi}F(1+\eta)^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi. \tag{A 13}$$

Combining (A 10) and (A 12), the identity (A 3) follows immediately. To establish (A 4), we rewrite (A 8) as

$$E_T = T\int_{S_{\mathrm{B}}}\mathrm{d}s_{\mathrm{B}} = T\int_{S_{\mathrm{B}}}\boldsymbol{n}_{\mathrm{B}}\cdot\boldsymbol{n}_{\mathrm{B}}\,\mathrm{d}s_{\mathrm{B}} = T\int_{V_{\mathrm{B}}}\nabla\cdot\boldsymbol{n}_{\mathrm{B}}\,\mathrm{d}^3\boldsymbol{x},$$

where the surface integral on the bubble surface is transferred into a volume integral over the bubble volume V_{B} through the divergence theorem. On taking the time derivative of this and using the transport theorem (A 9), we find that

$$\frac{\mathrm{d}E_T}{\mathrm{d}t} = T\int_{V_{\mathrm{B}}}\frac{\mathrm{D}}{\mathrm{D}t}(\nabla\cdot\boldsymbol{n}_{\mathrm{B}})\,\mathrm{d}^3\boldsymbol{x} = T\int_{V_{\mathrm{B}}}\nabla\cdot\left(\frac{\partial\boldsymbol{n}_{\mathrm{B}}}{\partial t}+\nabla\phi\,\nabla\cdot\boldsymbol{n}_{\mathrm{B}}\right)\mathrm{d}^3\boldsymbol{x},$$

which can be transferred back to surface integrals to give

$$\frac{\mathrm{d}E_T}{\mathrm{d}t} = T\int_{S_{\mathrm{B}}}\frac{\partial\boldsymbol{n}_{\mathrm{B}}}{\partial t}\cdot\boldsymbol{n}_{\mathrm{B}}\,\mathrm{d}s_{\mathrm{B}} + T\int_{S_{\mathrm{B}}}\nabla\cdot\boldsymbol{n}_{\mathrm{B}}(\nabla\phi\cdot\boldsymbol{n}_{\mathrm{B}})\,\mathrm{d}s_{\mathrm{B}}.$$

This is precisely (A 4) once it is recognized that the first term on the right-hand side vanishes because $\partial(\boldsymbol{n}_{\mathrm{B}}\cdot\boldsymbol{n}_{\mathrm{B}})/\partial t = 0$.

Now, we derive expressions for E_{K}, E_{W}, E_{B} and E_T in terms of the perturbation variables η and ϕ. Since $\nabla^2\phi = 0$, we have $(\nabla\phi)^2 = \nabla\cdot(\phi\nabla\phi)$, so that the volume integral in the definition (A 5) for the kinetic energy E_{K} can be transferred into surface integrals, which leads to

$$E_{\mathrm{K}} = \int_{S_\infty}\tfrac{1}{2}\phi\nabla\phi\cdot\boldsymbol{n}_\infty\,\mathrm{d}s_\infty - \int_{S_{\mathrm{B}}}\tfrac{1}{2}\phi\nabla\phi\cdot\boldsymbol{n}_{\mathrm{B}}\,\mathrm{d}s_{\mathrm{B}}.$$

The integral over the control surface S_∞ vanishes if S_∞ is chosen far away from the bubble because ϕ decays at least like $1/r$. The integral over the bubble surface can be simplified by using the identities (A 11) and (A 13), which yields

$$E_{\mathrm{K}} = \int_0^{\pi}\int_0^{2\pi}\tfrac{1}{2}\phi\frac{\partial\eta}{\partial t}(1+\eta)^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

where ϕ is to be evaluated at the bubble surface $r = 1+\eta$. Expanding the integrand by the Taylor expansion (2.9), retaining terms up to the order η^2 and using an overbar to denote the average over a unit sphere, it is straightforward to derive

$$E_K = -2\pi\,\overline{\phi\frac{\partial\eta}{\partial t}}. \tag{A 14}$$

The work done by p_∞ at infinity, defined by (A 6), is proportional to the mass flow through the control surface S_∞, which is equal to the mass flow through the bubble surface S_B from the mass conservation law (compressibility being neglected). Thus, the surface integral in (A 6) can actually be performed on the bubble surface, namely,

$$E_W = \int_0^t \int_{S_B} \nabla\phi\cdot \boldsymbol{n}_B\,\mathrm{d}s_B\,\mathrm{d}t.$$

The surface integral is now equal to $\mathrm{d}V_B/\mathrm{d}t$ according to (A 12), which can be used to further carry out the t-integral to yield

$$E_W = (V_B - V_0) = 4\pi(\overline{\eta}+\overline{\eta^2}), \tag{A 15}$$

where the last step follows from applying the Taylor expansion to V_B with the result truncated after the term proportional to η^2. The bubble internal energy E_B can be expressed in terms of the surface deformation η in a straightforward way because

$$p_B V_B = (1+2T)\,V_0\left(\frac{V_B}{V_0}\right)^{1-\gamma},$$

from the adiabatic law (2.6). Expanding V_B in terms of η, retaining terms up to the order η^2 and substituting the results into the definition (A 7), it is found that

$$E_B = \frac{(1+2T)\,V_0}{\gamma-1} - 4\pi(1+2T)\,(\overline{\eta}+\overline{\eta^2}-\tfrac{3}{2}\gamma\overline{\eta}^2). \tag{A 16}$$

Finally the surface tension energy E_T can be expressed in terms of η by noticing that

$$\int_{S_B} \mathrm{d}s_B = \int_0^\pi\int_0^{2\pi} (1+\eta)\,[(1+\eta)^2+(\nabla_s\,\eta)^2]^{\frac{1}{2}}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi.$$

Expanding the integrand in a power series of η, the result to the order η^2 gives

$$E_T = 4\pi T(1+2\overline{\eta}+\overline{\eta^2}+\tfrac{1}{2}\overline{(\nabla_s\,\eta)^2}). \tag{A 17}$$

It can be seen that (A 16) and (A 17) contain terms independent of η, which are the internal energy and the surface-tension energy of the bubble in its equilibrium state.

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